the line through **C** perpendicular to the direction of the slider and the line through **A** parallel this direction. Let the intersection of these two lines be the point **B**. The length $e = |\mathbf{B} - \mathbf{C}|$ is the joint offset, and $s = |\mathbf{A} - \mathbf{B}|$ is the slide distance of the prismatic joint. Denote the input crank angle by θ and let ψ be the angle measured about **C** to the segment **CB**.

These conventions allow us to introduce the intermediate parameters b and β given by

$$b = \sqrt{s^2 + e^2} \quad \text{and} \quad \tan\beta = \frac{s}{e}. \tag{2.40}$$

The cosine law for the triangle $\triangle\mathbf{COA}$ yields the relation

$$b^2 = g^2 + r^2 - 2rg\cos\theta. \tag{2.41}$$

Substitute $s^2 + e^2$ to obtain

$$s = \sqrt{g^2 + r^2 - e^2 - 2rg\cos\theta}. \tag{2.42}$$

This defines the joint slide s for a given crank angle θ. Notice that this equation can also be solved to determine θ for a given slide:

$$\cos\theta = \frac{g^2 + r^2 - e^2 - s^2}{2rg}. \tag{2.43}$$

This latter situation arises when the slider is the piston in a linear actuator driving the RR crank.

The rotation angle ψ of the RP crank is determined using the fact that the coordinates of the pivot **A** can be written in two ways

$$\mathbf{A} = \begin{Bmatrix} r\cos\theta \\ r\sin\theta \end{Bmatrix} = \begin{Bmatrix} g + b\cos(\psi+\beta) \\ b\sin(\psi+\beta) \end{Bmatrix}. \tag{2.44}$$

These equations yield the formula

$$\psi + \beta = \arctan\frac{r\sin\theta}{r\cos\theta - g}. \tag{2.45}$$

Notice that β is determined from s by (2.40).

The range of movement of the cranks and the sliding joint for this linkage can be analyzed in the same way as shown above for the RRRP linkage.

2.3 Position Analysis of the 4R Linkage

Given a planar 4R closed chain, we can identify an input RR crank and an output RR crank, Figure 2.5. Let the fixed and moving pivots of the input crank be **O** and **A**, respectively, and that the fixed and moving pivots of the output crank be **C** and

B. The distances between these points characterize the linkage:

$$a = |\mathbf{A} - \mathbf{O}|, b = |\mathbf{B} - \mathbf{C}|, g = |\mathbf{C} - \mathbf{O}|, h = |\mathbf{B} - \mathbf{A}|. \tag{2.46}$$

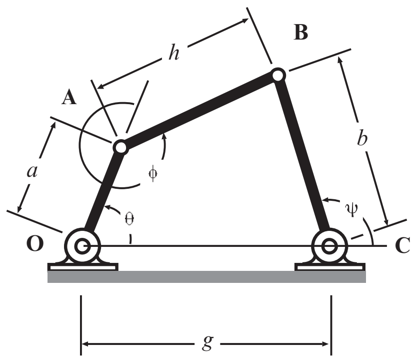

Fig. 2.5 The link lengths that define a 4R linkage.

To analyze the linkage, we locate the origin of the fixed frame F at **O**, and orient it so that the x-axis passes through the other fixed pivot **C**. Let θ be the input angle measured around **O** from the x-axis of F to **OA**. Similarly, let ψ be the angular position of the output crank **CB**.

2.3.1 Output Angle

The relationship between the input angle θ of the driving crank and the angle ψ of the driven crank is obtained from the condition that **A** and **B** remain a fixed distance apart throughout the motion of the linkage. Since $h = |\mathbf{B} - \mathbf{A}|$ is constant, we have the constraint equation

$$(\mathbf{B} - \mathbf{A}) \cdot (\mathbf{B} - \mathbf{A}) - h^2 = 0. \tag{2.47}$$

The coordinates of **A** and **B** in F are given by

$$\mathbf{A} = \begin{Bmatrix} a\cos\theta \\ a\sin\theta \end{Bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{Bmatrix} g + b\cos\psi \\ b\sin\psi \end{Bmatrix}. \tag{2.48}$$

Substitute these coordinates into (2.47) to obtain

$$b^2 + g^2 + 2gb\cos\psi + a^2 - 2(a\cos\theta(g + b\cos\psi) + ab\sin\theta\sin\psi) - h^2 = 0. \tag{2.49}$$

Gathering the coefficients of $\cos\psi$ and $\sin\psi$, we obtain the constraint equation for the 4R chain as

$$A(\theta)\cos\psi + B(\theta)\sin\psi = C(\theta), \tag{2.50}$$

where

$$\begin{aligned} A(\theta) &= 2ab\cos\theta - 2gb, \\ B(\theta) &= 2ab\sin\theta, \\ C(\theta) &= g^2 + b^2 + a^2 - h^2 - 2ag\cos\theta. \end{aligned} \tag{2.51}$$

The solution to this equation is

$$\psi(\theta) = \arctan\left(\frac{B}{A}\right) \pm \arccos\left(\frac{C}{\sqrt{A^2+B^2}}\right). \tag{2.52}$$

Equations of the form (2.50) arise many times in the analysis of linkages, so we present its solution in Appendix A for easy reference, see (A.1).

Notice that there are two angles ψ for each angle θ. This arises because the moving pivot **B** of the output crank can be assembled above or below the diagonal joining the moving pivot **A** of the input crank to the fixed pivot **C** of the output crank. The angle $\delta = \arctan(B/A)$ defines the location of this diagonal, and $\varepsilon = \arccos(C/\sqrt{A^2+B^2})$ is the angle above and below this diagonal that locates the output crank.

The argument of the arccosine function must be in the range -1 to $+1$, which places a solvability constraint on the coefficients A, B, and C. Specifically, for a solution to exist we must have

$$A^2 + B^2 - C^2 \geq 0. \tag{2.53}$$

If this constraint is not satisfied, then the linkage cannot be assembled for the specified input crank angle θ.

2.3.2 Coupler Angle

Let ϕ denote the angle of the coupler measured about **A** relative to the segment **OA**, so $\theta + \phi$ measures the angle to **AB** from the x-axis of F. The coordinates of **B** can also be defined in terms of ϕ as

$$\mathbf{B} = \begin{Bmatrix} a\cos\theta + h\cos(\theta+\phi) \\ a\sin\theta + h\sin(\theta+\phi) \end{Bmatrix}. \tag{2.54}$$

Equating the two forms for **B**, we obtain the loop equations of the four-bar linkage

$$\begin{aligned} a\cos\theta + h\cos(\theta+\phi) &= g + b\cos\psi, \\ a\sin\theta + h\sin(\theta+\phi) &= b\sin\psi. \end{aligned} \tag{2.55}$$

For a given value of the drive crank θ, determine ψ using (2.52) then $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$ are given by

$$\cos(\theta+\phi) = \frac{g + b\cos\psi - a\cos\theta}{h} \quad \text{and} \quad \sin(\theta+\phi) = \frac{b\sin\psi - a\sin\theta}{h}. \quad (2.56)$$

Thus, the value of the coupler angle is obtained as

$$\phi = \arctan\left(\frac{b\sin\psi - a\sin\theta}{g + b\cos\psi - a\cos\theta}\right) - \theta. \quad (2.57)$$

Notice that a unique value for ϕ is associated with each of the two solutions for the output angle ψ.

2.3.2.1 An Alternative Derivation

It is useful here to present a direct calculation of the coupler angle ϕ associated with a given crank angle θ. The derivation is identical to that above for the output angle. However, our standard frame is now F', positioned with its origin at **A** and its x-axis along the vector $\mathbf{O} - \mathbf{A}$. In this coordinate frame, the pivots **B** and **C** have the coordinates

$$^{F'}\mathbf{B} = \begin{Bmatrix} h\cos(\phi - \pi) \\ h\sin(\phi - \pi) \end{Bmatrix} \quad \text{and} \quad ^{F'}\mathbf{C} = \begin{Bmatrix} a + g\cos(\pi - \theta) \\ g\sin(\pi - \theta) \end{Bmatrix}. \quad (2.58)$$

The constraint $(\mathbf{B} - \mathbf{C}) \cdot (\mathbf{B} - \mathbf{C}) = b^2$ yields the equation

$$A(\theta)\cos\phi + B(\theta)\sin\phi = C(\theta), \quad (2.59)$$

where

$$\begin{aligned} A(\theta) &= 2ah - 2gh\cos\theta, \\ B(\theta) &= 2gh\sin\theta, \\ C(\theta) &= b^2 - a^2 - g^2 - h^2 + 2ag\cos\theta. \end{aligned} \quad (2.60)$$

This equation is solved in exactly the same way as before (A.1). It results in two values for ϕ for each crank angle θ. The output angle ψ associated with each of these coupler angles can be determined from the loop equations of the linkage written for **C** in F'.

This equation for the coupler angle is used in solutions for four and five position synthesis of a planar 4R linkage.

2.3.3 Transmission Angle

The angle ζ between the coupler and the driven crank at **B** is called the *transmission angle* of the linkage. If the only external loads on the linkage are torques on the input